THREE CONSECUTIVE ALMOST SQUARES

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ABSTRACT. Given a positive integer n, we let $\mathrm{sfp}(n)$ denote the squarefree part of n. We determine all positive integers n for which $\max\{\mathrm{sfp}(n), \mathrm{sfp}(n+1), \mathrm{sfp}(n+2)\} \leq 150$ by relating the problem to finding integral points on elliptic curves. We also prove that there are infinitely many n for which

$$\max\{\mathrm{sfp}(n), \mathrm{sfp}(n+1), \mathrm{sfp}(n+2)\} < n^{1/3}.$$

1. INTRODUCTION

The positive integers 48, 49 and 50 are consecutive, and are "almost squares", namely $48 = 3 \cdot 4^2$, $49 = 7^2$ and $50 = 2 \cdot 5^2$. Does this phenomenon ever occur again? That is, is there a positive integer $n > 48$ for which

$$\begin{aligned} n &= 3x^2 \\ n+1 &= y^2 \\ n+2 &= 2z^2 \end{aligned}$$

has an integer solution? The answer is no. One perspective on this is given by Cohen in Section 12.8.2 of [5], where this problem is solved using linear forms in logarithms. Another approach is to recognize that the pair of equations $2z^2 - y^2 = y^2 - 3x^2 = 1$ define an intersection of two quadrics in $\mathbb{P}^3$ and hence define a curve of genus 1. Siegel proved that there are only finitely many integer points on a genus 1 curve, and hence the system of equations above has only finitely many solutions. This method of solving simultaneous Pell equations is considered in [21].

We consider more generally the problem of, given integers a, b and c, finding integers n for which $n = ax^2$, $n+1 = by^2$ and $n+2 = cz^2$. Multiplying these equations gives $n(n+1)(n+2) = abc(xyz)^2$ and is related to the problem of finding consecutive integers that multiply to an "almost" perfect square. There is an extensive literature on related problems. On one hand, Erdős and Selfridge proved in [7] that no product of consecutive integers is a perfect power (a result generalized to arithmetic progressions by others, see [1] and [12]). In another direction, Győry proved [11] that the equation $n(n+1)\cdots(n+k-1) = bx^l$ has no integer solutions with $n > 0$ if the greatest prime factor of b does not exceed k.

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For a positive integer n, define $\mathrm{sfp}(n)$ to be the "squarefree part" of n, the smallest positive integer a so that $a|n$ and $\frac{n}{a}$ is a perfect square. In this paper, we consider positive integers n for which $\mathrm{sfp}(n)$, $\mathrm{sfp}(n+1)$ and $\mathrm{sfp}(n+2)$ are all small. Our first result is a classification of values of n for which $\mathrm{sfp}(n)$, $\mathrm{sfp}(n+1)$ and $\mathrm{sfp}(n+2)$ are all ≤ 150. We say that such an n is "non-trivial" if $\mathrm{sfp}(n) < n$, $\mathrm{sfp}(n+1) < n+1$ and $\mathrm{sfp}(n+2) < n+2$.

Theorem 1. *There are exactly* 25 *non-trivial* n *for which* $\mathrm{sfp}(n) \leq 150$*,* $\mathrm{sfp}(n+1) \leq 150$ *and* $\mathrm{sfp}(n+2) \leq 150$*, the largest of which is* $n = 9841094$*.*

Remark. *A table of all* 25 *values of* n *is given in Section 5.*

After this manuscript was complete, the authors were informed that the essence of this result follows from work of Michael Bennett and Gary Walsh (see [2]). Given positive integers a, b and c, a positive integer solution to $n = ax^2$, $n+1 = by^2$, $n+2 = cz^2$ yields $(n+1)^2 - n(n+2) = b^2y^4 - (ac)(xz)^2 = 1$. Bennett and Walsh prove that the system $b^2Y^4 - dX^2 = 1$ has at most one solution in positive integers. Moreover, if $T + U\sqrt{d}$ is a fundamental solution of Pell's equation $X^2 - dY^2 = 1$ define for $k \geq 1$, $T_k + U_k\sqrt{d} = (T + U\sqrt{d})^k$. Then $T_k = bx^2$ for some integer $x \in \mathbb{Z}$ for at most one k, and if any such k exists, then k is the smallest positive integer for which T_k is a multiple of b. Bennett and Walsh's approach uses linear forms in logarithms, while our approach relies heavily on the theory of elliptic curves. In particular, if C is the curve defined by the two equations $by^2 - ax^2 = 1, cz^2 - by^2 = 1$, then the Jacobian of C is isomorphic to $E : y^2 = x^3 - (abc)^2x$.

We rule out many of the 778688 candidates for the triple (a, b, c) by checking to see whether there are integer solutions to each of the three equations $by^2 - ax^2 = 1$, $cz^2 - b^2 = 1$ and $cz^2 - ax^2 = 2$. We also test C for local solvability, and use Tunnell's theorem (see [20]) to determine if the rank of E is positive. Finally, we use the surprising property that the natural map from C to E sends an integral solution on C to an integral point on E. It suffices therefore to compute all the integral points on E (using Magma [3]), which requires computing generators of the Mordell-Weil group. In many cases this is straightforward, but a number of cases require more involved methods (12-descent, computing the analytic rank, and the use of Heegner points).

Given the existence of large solutions, it is natural to ask how large $\max\{\mathrm{sfp}(n), \mathrm{sfp}(n+1), \mathrm{sfp}(n+2)\}$ can be as a function of n. This is the subject of our next result.

Theorem 2. *There are infinitely many positive integers* n *for which*

$$\max\{\mathrm{sfp}(n), \mathrm{sfp}(n+1), \mathrm{sfp}(n+2)\} < n^{1/3}.$$

Remark. *The following heuristic suggests that the exponent* $1/3$ *above is optimal. A partial summation argument shows that the number of positive integers* $n \leq x$ *with* $\mathrm{sfp}(n) \leq z$ *is* $\frac{12\sqrt{xz}}{\pi^2} + O(\sqrt{x}\log(z))$*. Assuming that* n*,* $n+1$ *and* $n+2$ *are "random" integers it follows that* $\mathrm{sfp}(n), \mathrm{sfp}(n+1)$ *and* $\mathrm{sfp}(n+2)$ *are all* $\leq n^{\alpha}$ *with probability about* $n^{3(\alpha-1)/2}$*. Therefore, the "expected" number of* n *for which* $\max\{\mathrm{sfp}(n), \mathrm{sfp}(n+1), \mathrm{sfp}(n+2)\} \leq n^{\alpha}$ *is infinite if* $3(\alpha-1)/2 \geq -1$ *and finite otherwise.*

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2. Background

We denote by $\mathbb{Q}_p$ the field of p-adic numbers. A necessary condition for a curve $X/\mathbb{Q}$ to have a rational solution is for it to have such a solution in $\mathbb{Q}_p$ for all primes p.

If d is an integer which is not a perfect square, let $\mathbb{Z}[\sqrt{d}] = \{a + b\sqrt{d} : a, b \in \mathbb{Z}\}$. This is a (not necessarily maximal) order in the quadratic field $\mathbb{Q}[\sqrt{d}]$. Let $N : \mathbb{Z}[\sqrt{d}] \to \mathbb{Z}$ be the norm map given by $N(a + b\sqrt{d}) = a^2 - db^2$.

We now describe some background about elliptic curves. For our purposes an elliptic curve is a curve of the shape

$$E : y^2 = x^3 + ax + b$$

where $a, b \in \mathbb{Q}$. Let $E(\mathbb{Q})$ be the set of pairs (x, y) of rationals numbers that solve the equation, together with the "point at infinity". This set has a binary operation on it: given P, Q in $E(\mathbb{Q})$, the line L through P and Q intersects E in a third point $R = (x, y)$. The point $P+Q$ is defined to be $(x, -y)$. This binary operation endows $E(\mathbb{Q})$ with the structure of an abelian group.

The Mordell-Weil theorem (see [17], Theorem VIII.4.1, for example) states the following.

Theorem 3. *For any elliptic curve $E/\mathbb{Q}$, the group $E(\mathbb{Q})$ is finitely generated. More, precisely,*

$$E(\mathbb{Q}) \cong E(\mathbb{Q})_{\text{tors}} \times \mathbb{Z}^{\text{rank}(E(\mathbb{Q}))},$$

where $E(\mathbb{Q})_{tors}$ is the (finite) torsion subgroup.

There is in general no algorithm which is proven to compute the rank of E, (see Section 3 of Rubin and Silverberg's paper [16]) but there are a number of procedures which work well in practice for relatively simple curves E. We will reduce the problem of solving $n = ax^2$, $n + 1 = by^2$ and $n + 2 = cz^2$ to finding points (X, Y) on the curve

$$E : Y^2 = X^3 - (abc)^2 X,$$

with $X, Y \in \mathbb{Z}$. A theorem of Siegel (see Theorem IX.4.3 of [17]) states that there are only finitely many points in $E(\mathbb{Q})$ with both coordinates integral. There are effective and practical algorithms (see [19], [9], [18] and [22]) to determine the set of integral points, provided the rank r can be computed and a set of generators for $E(\mathbb{Q})$ found. Given a point $P = (x, y) \in E(\mathbb{Q})$, the "naive height" of P is defined by writing $x = \frac{a}{b}$ with $a, b \in \mathbb{Z}$ with $\gcd(a, b) = 1$ and defining $h(P) = \log \max\{|a|, |b|\}$. The "canonical height" of P is defined to be

$$\hat{h}(P) = \lim_{n\to\infty} \frac{h(2^n P)}{4^n}.$$

The L-function of $E/\mathbb{Q}$ is defined to be

$$L(E,s)=\sum_{n=1}^{\infty}\frac{a_n(E)}{n^s}=\prod_p\left(1-a_p(E)p^{-s}+\varepsilon(p)p^{1-2s}\right)^{-1}$$

where $a_p(E)=p+1-\#E(\mathbb{F}_p)$, and

$$\varepsilon(p)=\begin{cases}1 & \text{if } p \text{ does not divide the conductor of } E,\\ 0 & \text{otherwise.}\end{cases}$$

The Birch and Swinnerton-Dyer conjecture states that $\operatorname{ord}_{s=1}L(E,s)=\operatorname{rank}(E(\mathbb{Q}))$, and moreover that

$$\lim_{s\to 1}\frac{L(E,s)}{(s-1)^r}=\frac{\Omega R(E/\mathbb{Q})Ш(E/\mathbb{Q})\prod_p c_p}{|T|^2}.$$

In our case, Ω is twice the real period, $R(E/\mathbb{Q})$ is the regulator of $E(\mathbb{Q})$ computed using the function $\hat{h}$ above, the c_p are the Tamagawa numbers, and $Ш(E/\mathbb{Q})$ is the Shafarevich-Tate group. The completed L-function $\Lambda(E,s)=N^{s/2}(2\pi)^{-s}\Gamma(s)L(E,s)$ satisfies the function equation $\Lambda(E,s)=w_E\Lambda(E,2-s)$, where w_E is the root number of E. Note that $w_E=1$ implies that $\operatorname{ord}_{s=1}L(E,s)$ is even, and $w_E=-1$ if $\operatorname{ord}_{s=1}L(E,s)$ is odd.

The best partial result in the direction of the Birch and Swinnerton-Dyer conjecture is the following.

Theorem 4 (Gross-Zagier [10], Kolyvagin [14], et al.)**.** *Suppose that $E/\mathbb{Q}$ is an elliptic curve and $\operatorname{ord}_{s=1}L(E,s)=0$ or 1. Then, $\operatorname{ord}_{s=1}L(E,s)=\operatorname{rank}(E(\mathbb{Q}))$.*

The work of Bump-Friedberg-Hoffstein [4] or Murty-Murty [15] is necessary to remove a condition imposed in the work of Gross-Zagier and Kolyvagin.

3. Proof of Theorem 2

Proof. Motivated by the observation that $8388223=127\cdot 257^2$ and $8388225=129\cdot 255^2$, we find a parametric family of solutions where $n=a\cdot b^2$ and $n+2=(a+2)\cdot(b-2)^2$.

If we write $(4+\sqrt{13})(649+180\sqrt{13})^m=x_m+y_m\sqrt{13}$ where x_m and y_m are integers, then $x_m^2-13y_m^2=3$ for all $m\geq 0$. We have that $x_0=4$, $x_1=4936$, $y_0=1$, $y_1=1369$ and

$$x_m=1298x_{m-1}-x_{m-2},y_m=1298y_{m-1}-y_{m-2}.$$

It is easy to see that x_m is periodic modulo 32 with period 8 and from this it follows that $x_{8m+7}\equiv 0 \pmod{32}$ for all $m\geq 0$. Set $a_m=x_{8m+7}/2$ and $n=4a_m^3-3a_m-1$. Then we have

$$\begin{aligned}n&=(a_m-1)(2a_m+1)^2,\\ n+1&=a_m(4a_m^2-3)=13a_my_{8m+7}^2,\text{ and}\\ n+2&=(a_m+1)(2a_m-1)^2.\end{aligned}$$

Since $16|a_m$, $\max\{\mathrm{sfp}(n), \mathrm{sfp}(n+1), \mathrm{sfp}(n+2)\} \leq \max\{a_m - 1, \frac{13a_m}{16}, a_m + 1\} = a_m + 1 < n^{1/3}$. $\square$

Remark. *The polynomial $4x^3 - 3x$ used in the proof above is the Chebyshev polynomial $T_3(x)$. This explains why $T_3(x) - 1$ and $T_3(x) + 1$ both have a double zero.*

4. Proof of Theorem 1

Since there are 92 squarefree integers ≤ 150, there are $778688 = 92^3$ possibilities for the triple (a, b, c). We first test four things before searching for integral points on $E : y^2 = x^3 - (abc)^2 x$. Suppose that $n = ax^2$, $n + 1 = by^2$ and $n + 2 = cz^2$ is an integral solution to the system of equations

$$by^2 - ax^2 = 1, \tag{1}$$

$$cz^2 - by^2 = 1, \tag{2}$$

$$cz^2 - ax^2 = 2. \tag{3}$$

4.1. **Greatest common divisor conditions.** If (x, y, z) is an integer solution to (1), (2) and (3), then $\gcd(a, b) = 1$, $\gcd(b, c) = 1$ and $\gcd(a, c) = 1$ or 2. This reduces the number of triples to consider to 425639.

4.2. **Norm equations.** We next check that each of the equations $by^2 - ax^2 = 1$, $cz^2 - by^2 = 1$, and $cz^2 - ax^2 = 2$ has an integer solution. The equation $by^2 - ax^2 = 1$ has an integer solution if and only if $Y^2 - abx^2 = b$ has an integer solution. This equation has a solution if and only if $\mathbb{Z}[\sqrt{ab}]$ has an element of norm b. We use Magma's routine `NormEquation` to test this. After these tests have been made, there are 2188 possibilities for (a, b, c) that remain.

4.3. **Local solvability of C.** Let $C \subseteq \mathbb{P}^3$ be the curve defined by the two equations $by^2 - ax^2 = w^2$, $cz^2 - by^2 = w^2$. This curve must have a rational point on it in order for there to be a non-trivial solution. We check whether C has points $(x : y : z : w)$ in $\mathbb{Q}_p$ for all primes p dividing $2abc$. This is done via Magma's `IsLocallySolvable` routine. This eliminates 244 possibilities.

4.4. **Rank of the elliptic curve E.** Let $E : y^2 = x^3 - (abc)^2 x$. Define a map M from non-trivial solutions to C, represented in the form $n = ax^2$, $n+1 = by^2$ and $n+2 = cz^2$ to E, given by

$$M(x, y, z, n) = ((n + 1)(abc), (abc)^2 xyz).$$

If $n > 0$, then $xyz > 0$ and so $M(x, y, z, n) \in E(\mathbb{Q})$ is an integral point, and one with a non-zero y-coordinate. We now use the following results about the family of elliptic curves E we consider.

Definition 5. *A natural number N is called* congruent *if there exists a right triangle with all three sides rational and area N.*

Consider the elliptic curve E over $\mathbb{Q}$ given by:

$$E(\mathbb{Q}) : y^2 = x^3 - N^2 x.$$

We have the following result.

Theorem 6 (Proposition I.9.18 of [13]). *The number N is congruent if and only if the rank of E is positive.*

Proposition I.9.17 of [13] implies that the only points of finite order on E are $(0,0)$, $(\pm N, 0)$ and the point at infinity. If for some $n > 0$ we have $n = ax^2$, $n+1 = by^2$ and $n+2 = cz^2$, then the point $M(x,y,z,n)$ on $E : Y^2 = X^3 - (abc)^2 X$ is, according to the above, a non-torsion point, and hence E has positive rank, and consequently, by Theorem 6, abc is a congruent number.

Theorem 7 (Tunnell 1983). *If N is squarefree and odd, and*

$$\#\left\{x,y,z \in \mathbb{Z} | N = 2x^2 + y^2 + 32z^2\right\} \neq \frac{1}{2}\#\left\{x,y,z \in \mathbb{Z} | N = 2x^2 + y^2 + 8z^2\right\},$$

then N is not congruent. If N is squarefree and even, and

$$\#\left\{x,y,z \in \mathbb{Z} : \frac{N}{2} = 4x^2 + y^2 + 32z^2\right\} \neq \frac{1}{2}\#\left\{x,y,z \in \mathbb{Z} : \frac{N}{2} = 4x^2 + y^2 + 8z^2\right\},$$

then N is not congruent.

To use Tunnell's theorem, we compute the generating function for the number of representations of N by $2x^2 + y^2 + 8z^2$ as

$$\sum_{x,y,z \in \mathbb{Z}} q^{2x^2+y^2+8z^2} = \left(\sum_{x \in \mathbb{Z}} q^{2x^2}\right)\left(\sum_{y \in \mathbb{Z}} q^{y^2}\right)\left(\sum_{z \in \mathbb{Z}} q^{8z^2}\right),$$

as well as the representations of N by $2x^2 + y^2 + 32z^2$, $4x^2 + y^2 + 8z^2$ and $4x^2 + y^2 + 32z^2$. Actually, for our purposes, it suffices to compute

$$\sum_{x=-X}^{X} \sum_{y=-Y}^{Y} \sum_{z=-Z}^{Z} q^{2x^2+y^2+8z^2},$$

where $X = \lfloor\sqrt{150^3/2}\rfloor$, $Y = \lfloor\sqrt{150^3}\rfloor$, and $Z = \lfloor\sqrt{150^3/8}\rfloor$.

The use of Tunnell's theorem rules out 530 of the 1944 remaining cases. In the case that the hypothesis of Tunnell's theorem is false, the Birch and Swinnerton-Dyer conjecture predicts that $E(\mathbb{Q})$ does have positive rank. In this case, we proceed to the next step.

4.5. **Computing integral points.** Once the program determines that the elliptic curve $E(\mathbb{Q}) : Y^2 = X^3 - (abc)^2 X$ has positive rank, we let Magma attempt to compute the integral points on the curve, using the routine `IntegralPoints`, which is based on the method developed in [9] and [19]. If this routine does not succeed within 15 minutes, which is sufficient time to perform a 4-descent, we abort the computation.

In 1377 of the 1414 cases that remain, Magma is able to determine the Mordell-Weil group and determine all of the integral points within 15 minutes. Once the integral points are determined, we check to see if they are in the image of the map $M(x, y, z, n)$ and if so, whether they correspond to a non-trivial solution.

There are 37 more difficult cases that remain, and in each case we are able to use other methods to determine the Mordell-Weil group. A table of these cases and the generators of the Mordell-Weil group are given at the page `http://users.wfu.edu/rouseja/MWgens.html`.

Of the 37 cases, there are four cases with root number -1 and rank ≤ 3 for which one point of infinite order has low height. In these, we numerically compute $L'(E, 1)$ and show that it is nonzero. Theorem 4 proves the rank is one in these cases. One of these cases is $a = 139$, $b = 89$ and $c = 109$. In this case, E has conductor $\approx 5.82 \cdot 10^{13}$ and computing $L'(E, 1)$ takes about 5 hours.

In the remaining 33 cases, rather than use the 8-descent (the default if the `IntegralPoints` command were to keep running), we use the 12-descent routines in Magma (due to Tom Fisher [8]) to find points of large height. The curve with $a = 137$, $b = 109$ and $c = 101$ has a generator with canonical height 1234, which is found after searching on the 12-covers for about 90 minutes. This approach is successful in all but one case.

4.6. **Heegner points.** All but one of the 778688 original cases are handled by the methods of the previous sections. The remaining case is $a = 67$, $b = 131$ and $c = 109$. The curve E has root number -1, rank ≤ 1 and conductor $\approx 2.9 \cdot 10^{13}$. A long computation shows that $L'(E, 1) \approx 72.604$. This suggests, assuming the Birch and Swinnerton-Dyer conjecture is true and $Ш(E/\mathbb{Q})$ is trivial, that a generator of the Mordell-Weil group has canonical height about 1692. Searching for points on the 12-covers up to a height of $3^{42} \cdot 10^5 \approx 10^{25}$ does not succeed in finding points.

For this final case, we use the method of Elkies described in [6]. This is a variant of the usual Heegner point method and is quite fast on quadratic twists of curves with low conductor. The modular curve $X_0(32)$ parametrizes pairs (E, C), where E is an elliptic curve, and $C \subseteq E$ is a cyclic subgroup of order 32. It is well-known that $X_0(32)$ is isomorphic to $y^2 = x^3 + 4x$, which in turn has a degree 2 map to $y^2 = x^3 - x$. Finding a rational point on $y^2 = x^3 - D^2 x$ is equivalent to finding a point on $y^2 = x^3 - x$ with x and $y/\sqrt{-D}$ both rational. If E is an elliptic curve with complex multiplication whose endomorphism ring $\mathcal{O}$ contains an ideal I so that $\mathcal{O}/I \cong \mathbb{Z}/32\mathbb{Z}$, this naturally gives rise to a point on $X_0(32)$, which is defined in the ring class field of $\mathcal{O}$. Taking the trace of such a point (in the Mordell-Weil group) gives rise to a point on $X_0(32)$ over an imaginary quadratic field.

In [6], Elkies gives a procedure for constructing rational points on $y^2 = x^3 - D^2 x$ with $D \equiv 7 \pmod 8$. We must handle one case with $D \equiv 5 \pmod 8$. We let $\mathcal{O} = \mathbb{Z}[4\sqrt{-D}]$. This ring has an ideal $I = \langle 32, 4 + 4\sqrt{-D} \rangle$ and $\mathcal{O}/I \cong \mathbb{Z}/32\mathbb{Z}$. We take a representative ideal J for each element of the class group of $\mathcal{O}$. Thinking of J as a lattice, $\mathbb{C}/IJ$ is an elliptic curve, and I/IJ is a subgroup of order 32 on the curve. Thus, $(\mathbb{C}/IJ, I/IJ)$ is a Heegner point on $X_0(32)$.

Via the isomorphism with $E : y^2 = x^3 + 4x$, we obtain a collection of points. Adding these points together on E gives a point defined in $\mathbb{Q}(\sqrt{-D})$, and eventually a rational point on $y^2 = x^3 - D^2x$. Running this computation in several simpler cases suggests that the resulting point P on $y^2 = x^3 - D^2x$ satisfies $mQ = P$ for some rational point Q, where m is equal to the number of divisors of D. Any point Q must have an x-coordinate of the form $x = em^2/n^2$, where e is a divisor of $2D$. We use this method to minimize the amount of decimal precision needed to compute P.

For $D = 67\cdot 131\cdot 109$, the class number of $\mathcal{O}$ is 3712. Given that we expect a point with canonical height 1692, we compute the Heegner points and their trace in the manner described in [6] using 850 digits of decimal precision. The result is a point Q with canonical height 1692.698, and the computation takes about 18 minutes. This point is a generator of the Mordell-Weil group of $E : y^2 = x^3 - D^2x$. Once the Mordell-Weil group is found, we check that there are no integral points on E. This completes the proof of Theorem 1.

5. Table of n

The following is a table of all 25 positive integers n with $\mathrm{sfp}(k) < \max\{k, 150\}$ for $k = n$, $n+1$ and $n+2$.

n	$\mathrm{sfp}(n)$	$\mathrm{sfp}(n+1)$	$\mathrm{sfp}(n+2)$
48	3	1	2
98	2	11	1
124	31	5	14
242	2	3	61
243	3	61	5
342	38	7	86
350	14	39	22
423	47	106	17
475	19	119	53
548	137	61	22
845	5	94	7
846	94	7	53
1024	1	41	114
1375	55	86	17
1519	31	95	1
1680	105	1	2
3724	19	149	46
9800	2	1	58
31211	59	3	13
32798	62	39	82
118579	19	5	141
629693	53	46	55
1294298	122	19	7
8388223	127	26	129
9841094	134	55	34

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